$$Sume\ \ combinatoriale - 2 -$$

$(a+b)^n = C_n^0 a^n + C_n^1 a^{n-1} b + C_n^2 a^{n-2} b^2 + \cdots + C_n^n b^n \;, n \in \mathbb{N}$

$C_n^k = C_n^{n-k} \;, 0 \le k \le n$ adică $C_n^0 = C_n^n \;, C_n^1 = C_n^{n-1} \;, C_n^2 = C_n^{n-2}$

$C_n^k = \frac{n}{k} C_{n-1}^{k-1} \quad \rightarrow \quad \frac{1}{k} C_{n-1}^{k-1} = \frac{1}{n} C_n^k$

$C_n^0 + C_n^1 + C_n^2 + \cdots + C_n^n = 2^n$

$C_n^0 - C_n^1 + C_n^2 - C_n^3 + \cdots + (-1)^n C_n^n = 0$

$C_n^0 + C_n^2 + C_n^4 + \cdots = 2^{n-1} = C_n^1 + C_n^3 + C_n^5 + \cdots$

$C_n^1 + 2C_n^2 + 3C_n^3 + \cdots + nC_n^n = n2^{n-1}$

Calculați:

1. $C_n^1 + 4C_n^2 + 7C_n^3 + \cdots + (3n-2)C_n^n$

$(3\cdot 1 - 2)C_n^1 + (3\cdot 2 - 2)C_n^2 + (3\cdot 3 - 2)C_n^3 + \cdots + (3n-2)C_n^n =$

$= 3(C_n^1 + 2C_n^2 + 3C_n^3 + \cdots + nC_n^n) - 2(C_n^1 + C_n^2 + \cdots + C_n^n) = 3n2^{n-1} - 2(2^n - 1) =$

$= (3n-4)2^{n-1} + 2$

2. $C_n^0 + 3C_n^1 + 5C_n^2 + \cdots + (2n+1)C_n^n$

3. $kC_n^1 + (k+2)C_n^2 + \cdots + (k+2n-2)C_n^n$

$$kC_n^1 + (k+2)C_n^2 + \cdots + (k+2n-2)C_n^n = \sum_{i=1}^{n} (k-2+2i)C_n^i =$$

$$= (k-2)\sum_{i=1}^{n} C_n^i + 2\sum_{i=1}^{n} iC_n^i = (k-2)(2^n-1) + 2n2^{n-1} = (k+n-2)2^n - k + 2$$

4. $C_n^0 + \frac{1}{2}C_n^1 + \frac{1}{3}C_n^2 + \cdots + \frac{1}{n+1}C_n^n$

$\frac{1}{k+1}C_n^k = \frac{1}{n+1}C_{n+1}^{k+1} \quad \rightarrow \quad \frac{1}{n+1}C_{n+1}^1 + \frac{1}{n+1}C_{n+1}^2 + \frac{1}{n+1}C_{n+1}^3 + \cdots + \frac{1}{n+1}C_{n+1}^{n+1} =$

$\frac{1}{n+1}(C_{n+1}^1 + C_{n+1}^2 + \cdots + C_{n+1}^{n+1}) = \frac{1}{n+1}(2^{n+1} - 1)$

5. $\frac{C_n^0}{2} + \frac{C_n^1}{3} + \frac{C_n^2}{4} + \cdots + \frac{C_n^n}{n+2}$

$\frac{1}{n+2}C_{n+2}^{k+2} = \frac{1}{k+2}C_{n+1}^{k+1} = \frac{1}{k+2}\frac{n+1}{k+1}C_n^k \quad \rightarrow \quad \frac{C_n^k}{k+2} = \frac{k+1}{(n+1)(n+2)}C_{n+2}^{k+2}$

$\frac{C_n^0}{2} + \frac{C_n^1}{3} + \frac{C_n^2}{4} + \cdots + \frac{C_n^n}{n+2} = \frac{1}{(n+1)(n+2)}(C_{n+2}^2 + 2C_{n+2}^3 + 3C_{n+2}^4 + \cdots + (n+1)C_{n+2}^{n+2})$

$= \frac{1}{(n+1)(n+2)}\left(C_{n+2}^1 + 2C_{n+2}^2 + \cdots + (n+2)C_{n+2}^{n+2} - (C_{n+2}^1 + C_{n+2}^2 + \cdots + C_{n+2}^{n+2})\right) =$

$= \frac{1}{(n+1)(n+2)}\left((n+2)2^{n+1} - (2^{n+2} - 1)\right) = \frac{1}{(n+1)(n+2)}(n2^{n+1} + 1)$

6. $\frac{C_n^0}{1\cdot 2} + \frac{C_n^1}{2\cdot 3} + \frac{C_n^2}{3\cdot 4} + \cdots + \frac{C_n^n}{(n+1)(n+2)}$

7. $C_n^0 - \frac{1}{2}C_n^1 + \frac{1}{3}C_n^2 - \cdots + (-1)^n \frac{1}{n+1}C_n^n$