Şiruri de numere reale

Şirul (a_n) este mărginit dacă mulţimea termenilor lui este mărginită.
$\exists\, a,b \in \mathbb{R}, a<b$, a.î. $a_n \in [\,a,b\,], \forall n \in \mathbb{N}$
Ex. $(a_n)_{n\ge1},\ a_n=\frac{2n-1}{n} \Rightarrow a_n \in [1,2]$

Dacă
$\begin{cases} a_{n+1}-a_n>0 \text{ se spune că şirul } (a_n) \text{ este strict crescător.} \\ a_{n+1}-a_n<0 \text{ se spune că şirul } (a_n) \text{ este strict descrescător.}\end{cases}$
Sau dacă
$\begin{cases} \frac{a_{n+1}}{a_n}>1 \text{ se spune că şirul } (a_n) \text{ este strict crescător.} \\ \frac{a_{n+1}}{a_n}<1 \text{ se spune că şirul } (a_n) \text{ este strict descrescător.}\end{cases}$
Ex. $(a_n)_{n\ge1},\ a_n=\frac{2n-1}{n}$. *Studiaţi monotonia şirului.*

$$a_{n+1}-a_n=\frac{2(n+1)-1}{n+1}-\frac{2n-1}{n}=\frac{2n+1}{n+1}-\frac{2n-1}{n}=\frac{1}{n(n+1)}>0$$

$\Rightarrow (a_n)_{n\ge1}$ *este strict crescător*

Un şir care are limita finită se numeşte şir convergent.
Un şir care nu este convergent este divergent.

$\lim\limits_{n\to\infty}\frac{1}{n}=0 \Rightarrow (a_n)_{n\ge1}, a_n=\frac{1}{n}$ *este convergent*

$\lim\limits_{n\to\infty} n=\infty \Rightarrow (a_n)_{n\ge1},\ a_n = n$ *este divergent*

$\nexists \lim\limits_{n\to\infty}(-2)^n \Rightarrow (a_n)_{n\ge1},\ a_n=(-2)^n$ *este divergent*

1) $\lim\limits_{n\to\infty} n^k=\begin{cases}\infty, k>0\\ 1, k=0\\ 0, k<0\end{cases}$

2) $\lim\limits_{n\to\infty}(a_kn^k+\dots+a_1n+a_0)=\lim\limits_{n\to\infty}a_kn^k=\begin{cases}\infty, & a_k>0\\ -\infty, & a_k<0\end{cases}$

3) $\lim\limits_{n\to\infty}\frac{a_kn^k+\dots+a_1n+a_0}{b_ln^l+\dots+b_1n+b_0}=\lim\limits_{n\to\infty}\frac{a_kn^k}{b_ln^l}=\begin{cases}\frac{a_k}{b_l}\cdot\infty, k>l\\ \frac{a_k}{b_l}, k=l\\ 0, k<l\end{cases}$

4) $\lim\limits_{n\to\infty}a^n=\begin{cases}\infty, a>1\\ 1, a=1\\ 0, |a|<1\\ \nexists, a\le -1\end{cases}$

5) $\lim\limits_{n\to\infty}n^ka^n=0, k\in \mathbb{N}^*, a\in(-1,1)\leftrightarrow |a|<1$

6) $\lim\limits_{n\to\infty}\frac{a^n}{n^k}=\infty, k\in\mathbb{N}, a>1$

7) $\lim\limits_{n\to\infty}\frac{\ln n}{n}=0$

8) $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e,\qquad e=2{,}71\dots$ *numărul lui Euler*

1. $\lim\limits_{n\to\infty}n^3=\infty$

2. $\lim\limits_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}=\lim\limits_{n\to\infty}\frac{1}{2\sqrt{n}}=0$

3. $\lim\limits_{n\to\infty}\frac{1+2+\dots+n}{n^2}=\frac{\infty}{\infty}=\lim\limits_{n\to\infty}\frac{n(n+1)}{2n^2}=\frac{1}{2}$

4. $\lim\limits_{n\to\infty}\left(\sqrt{n^2+5n+1}-n\right)=\infty-\infty=$
$=\lim\limits_{n\to\infty}\frac{n^2+5n+1-n^2}{\sqrt{n^2+5n+1}+n}=\lim\limits_{n\to\infty}\frac{5n}{2n}=\frac{5}{2}$

5. $\lim\limits_{n\to\infty}\left(\sqrt[3]{n^3+5n^2+1}-n\right)=\infty-\infty=$
$=\lim\limits_{n\to\infty}\frac{n^3+5n^2+1-n^3}{\sqrt[3]{(n^3+5n^2+1)^2}+n\sqrt[3]{n^3+5n^2+1}+n^2}=$
$=\lim\limits_{n\to\infty}\frac{5n^2}{3n^2}=\frac{5}{3}$

6. $\lim\limits_{n\to\infty}\left(\frac{n}{n+1}\right)^n=1^\infty=\lim\limits_{n\to\infty}\left(1+\frac{n}{n+1}-1\right)^n=$
$=\lim\limits_{n\to\infty}\left(1+\frac{-1}{n+1}\right)^n=$
$=\lim\limits_{n\to\infty}\left[\left(1+\frac{-1}{n+1}\right)^{\frac{n+1}{-1}}\right]^{\frac{-1}{n+1}n}=e^{\lim\limits_{n\to\infty}\frac{-n}{n+1}}=e^{-1}$

7. $\lim\limits_{n\to\infty}(1+2^n+3^n)=\lim\limits_{n\to\infty}3^n=\infty,\qquad 3>1$

Limite remarcabile

1) $\lim\limits_{n\to\infty}\frac{\sin x_n}{x_n}=1, x_n\in\mathbb{R}^*, x_n\to0$

2) $\lim\limits_{n\to\infty}\frac{tgx_n}{x_n}=1, x_n\in\mathbb{R}^*, x_n\to0$

3) $\lim\limits_{n\to\infty}\frac{\arcsin x_n}{x_n}=1, x_n\in\mathbb{R}^*, x_n\to0$

4) $\lim\limits_{n\to\infty}\frac{arctgx_n}{x_n}=1, x_n\in\mathbb{R}^*, x_n\to0$

5) $\lim\limits_{n\to\infty}\frac{\ln(1+x_n)}{x_n}=1, x_n\in\mathbb{R}^*, x_n\to0, 1+x_n>0$

6) $\lim\limits_{n\to\infty}\frac{a^{x_n}-1}{x_n}=\ln a, x_n\in\mathbb{R}^*, x_n\to0, a>0, a\ne1$

7) $\lim\limits_{n\to\infty}\frac{(1+x_n)^r-1}{x_n}=r, x_n\in\mathbb{R}^*, x_n\to0, r\in\mathbb{R}$

1. $\lim\limits_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}\right)=$
utilizăm $\lim\limits_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\ln n\right)=c,$
$c=0{,}57\dots$ *constanta lui Euler*
$=\lim\limits_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\ln n+\ln n\right)=\lim\limits_{n\to\infty}(c+\ln n)=\infty$

2. $\lim\limits_{n\to\infty}\frac{1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}}{n}=\frac{\infty}{\infty}=Stolz-Cesaro=\lim\limits_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}=$
$=\lim\limits_{n\to\infty}\frac{1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}+\frac{1}{n+1}-\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}\right)}{n+1-n}=$
$=\lim\limits_{n\to\infty}\frac{1}{n+1}=0$

3. $\lim\limits_{n\to\infty}\frac{5^n}{n!}$
utilizăm Criteriul raportului
$\lim\limits_{n\to\infty}\frac{\frac{5^{n+1}}{(n+1)!}}{\frac{5^n}{n!}}=\lim\limits_{n\to\infty}\frac{5}{n+1}=0<1\Rightarrow\lim\limits_{n\to\infty}\frac{5^n}{n!}=0$

4. $\lim\limits_{n\to\infty}\sqrt[n]{n^2+n+1}$
utilizăm Criteriul radicalului
$\lim\limits_{n\to\infty}\frac{(n+1)^2+(n+1)+1}{n^2+n+1}=\lim\limits_{n\to\infty}\frac{n^2}{n^2}=1\Rightarrow\ \lim\limits_{n\to\infty}\sqrt[n]{n^2+n+1}=1$

5. $\lim\limits_{n\to\infty}\frac{\sin n}{n}=0$
utilizând Criteriul de convergenţă
$a_n=\sin n\in[-1,1]$ *şi* $b_n=\frac{1}{n}\to0$